Eur. Phys. J. B 8, 137–141 (1999)

THE EUROPEAN PHYSICAL JOURNAL B

Effect of parametric modulation on pattern formation in reaction diffusion systems

A. Bhattacharyay[a] and J.K. Bhattacharjee

Department of Theoretical Physics, Indian Association for the Cultivation of Science, Jadavpur, Calcutta 700 032, India

Received 11 March 1998

Abstract. Reaction diffusion systems can exhibit both spatial and temporal patterns. We show that the effect of spatial variation of the removal rate can have significant effect on the stability boundaries. In particular there can be a case of parametric resonance.

PACS. 87.10.+e General, theoretical, and mathematical biophysics (including logic of biosystems, quantum biology, and relevant aspects of thermodynamics, information theory, cybernetics, and bionics) – 47.70.Fw Chemically reactive flows

1 Introduction

The "non common sensical" result that two species with widely differing diffusivity can produce stable inhomogenous patterns ("non common sensical" because diffusion normally smooths out inhomogeneities) was first obtained by Turing. Today it is widely accepted that in the pattern formation first studied by Turing [1], there are two central features [2,3] (a) local self-enhancement and (b) long-range inhibition. The self enhancement is necessary for the amplification of small local inhomogeneities but is not sufficient to generate stable patterns. With this in mind Gierer and Meinhardt introduced the two species (A and B are two species concentration) model.

$$\frac{\partial A}{\partial t} = D_A \bigtriangledown^2 A + \rho_A \left(\frac{A^2}{1+k_a^2}\right)\frac{1}{B} - \mu_A A + \sigma_A \quad (1)$$

$$\frac{\partial B}{\partial t} = D_B \bigtriangledown^2 B + \rho_B A^2 - \mu_B B + \sigma_B. \quad (2)$$

The species B is an antagonist and consequently $D_B \gg D_A$. The coefficients μ_A and μ_B are the removal rates, the production terms are given by σ_A and σ_B, the cross reaction coefficients are given by ρ_A and ρ_B. The constant k_a is called a saturation constant. The natural pattern formation parameter range require $\mu_A \ll \mu_B$, as otherwise the local amplification of A would not be very effective. The constant k_a is not necessary for pattern formation (although it has a strong impact on the actual shape) and in what follows, we will set $k_a = 0$. Similarly the self production rate of B is unimportant for pattern formation, its suffices to have a cross reaction rate ρ_B which produces the species B. Thus, we may set $\sigma_B = 0$. The concentrations A and B may be rescaled to set $\rho_A = \mu_A$ and $\rho_B = \mu_B$. We can also rescale space and time appropriately and finally arrive at the governing equations [4]

$$\frac{\partial A}{\partial t} = D \bigtriangledown^2 A + \frac{A^2}{B} - A + \sigma \quad (3)$$

$$\frac{\partial B}{\partial t} = \bigtriangledown^2 B + \mu(A^2 - B). \quad (4)$$

The above model has three parameters D, σ and μ. The stability characteristics of it have been well investigated showing the possibility of a time dependent spatially periodic state (spatial pattern) and a spatially homogeneous oscillatory state (temporal pattern) in addition to the homogeneous steady state. The stability boundaries have been investigated.

In this work, we investigate the above model to include spatial variation in the reaction parameters μ and σ. Unlike the diffusion coefficients, the parameters are going to be sensitive to the environment and either spontaneously are or can be induced to be functions of the spatial coordinate. However, a spatially varying σ precludes the possibility of a homogeneous steady state and hence will not be considered here. We consider a spatially varying μ, which always allows for a homogeneous steady state.Such parametric variations are common in the field of convective instabilities in fluids [5–8]. The variation is taken to be periodic with wave number K, so that μ is replaced by $\bar{\mu_0}(1+\epsilon\cos Kx)$ in equations (3, 4). We consider spatial variation in μ, in order to keep the homogeneous steady (the primary fixed point) intact in the presence of a modulation. This helps remove mathematical complications. In general, systems would be such that only μ_B in equation (2) would get modulated with ρ_b unaffected. This would imply that instead of a homogeneous steady state we would have an almost homogeneous (for small

[a] e-mail: arijit@hp2.saha.ernet.in

modulations) steady state. This complication in the description of the primary fixed point leads to proliferation of terms with no new qualitative insight. Consequently we have modulation on μ of equation (4). The amplitude ϵ is taken to be small so that perturbation theory remains valid. Our main observation is the possibility of a parametric resonance. This increases the likelihood of pattern formation in the system. In the absence of the resonance, there is also a stabilization of the patterned state. Thus, our conclusion is that an oscillatory spatial variation in the parameter μ stabilizes the spatial pattern. In Section 2, we rederive the standard results for constant μ and in Section 3 the results for an oscillatory μ are produced. Section 4 consists of a brief summary.

2 The standard model

In this section, we derive the standard results for the model represented by equations (3, 4) in order to make our subsequent calculations with the varying parameter more transparent. For the homogeneous steady state of equations (3, 4) we have $\dot{A} = \dot{B} = \nabla^2 A = \nabla^2 B = 0$, leading to

$$A = 1 + \sigma \tag{5}$$

$$B = A^2 = (1+\sigma)^2. \tag{6}$$

Considering perturbations δA and δB about the above solution and linearizing in these variations

$$L \begin{pmatrix} \delta A \\ \delta B \end{pmatrix} = 0 \tag{7}$$

where

$$L = \begin{pmatrix} \frac{\partial}{\partial t} - D\nabla^2 - \frac{1-\sigma}{1+\sigma} & \frac{1}{(1+\sigma)^2} \\ 2\mu(1+\sigma) & \nabla^2 - \frac{\partial}{\partial t} - \mu \end{pmatrix}. \tag{8}$$

If the solutions for δA and δB have the structure $\exp(pt)\cos(kx)$, then

$$\begin{pmatrix} p + Dk^2 - \frac{1-\sigma}{1+\sigma} & \frac{1}{(1+\sigma)^2} \\ 2\mu(1+\sigma) & -k^2 - p - \mu \end{pmatrix} \begin{pmatrix} \delta A_0 \\ \delta B_0 \end{pmatrix} = 0 \tag{9}$$

and the solvability criterion is

$$p^2 + \alpha p + \beta = 0 \tag{10}$$

where

$$\alpha = k^2(1+D) + \mu - \frac{1-\sigma}{1+\sigma} \tag{11}$$

and

$$\beta = Dk^4 + \left(\mu D - \frac{1-\sigma}{1+\sigma}\right) k^2 + \mu. \tag{12}$$

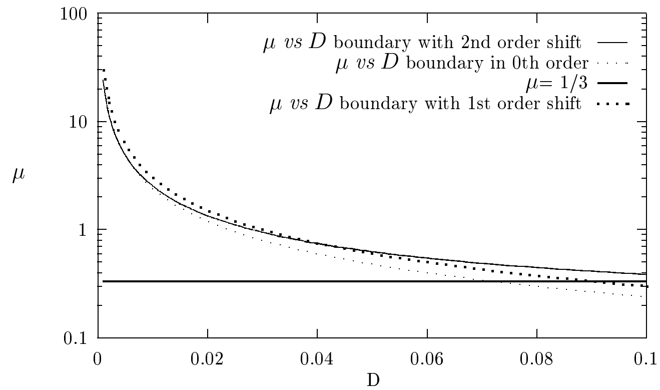

Fig. 1. The shifted μ *vs.* D phase boundaries in first and second order for $\epsilon = 1/2$ along with the boundaries in zeroth order.

From equation (10) it follows that the growth rate p is given by

$$2p = -\alpha \pm \sqrt{\alpha^2 - 4\beta}. \tag{13}$$

The homogeneous steady state is unstable if $\mathrm{Re}(p) > 0$. If $\beta < 0$, then this instability will certainly occur. For $\alpha^2 - 4\beta > 0$ and $\beta > 0$, the growth rate is negative and the homogeneous steady state is stable. At $\beta = 0$ the state loses its stability to a time dependent inhomogeneous state, characterized by a wave-number k which from equation (12) is obtained as

$$2Dk^2 = \frac{1-\sigma}{1+\sigma} - \mu D \pm \sqrt{\left[\frac{1-\sigma}{1+\sigma} - \mu D\right]^2 - 4\mu D}. \tag{14}$$

That k is real is assured if

$$\left(\frac{1-\sigma}{1+\sigma} - \mu D\right)^2 \geq 4\mu D \tag{15}$$

or equivalently

$$\left(\sqrt{\frac{2}{1+\sigma}} - 1\right)^2 \geq \mu D. \tag{16}$$

For a given σ, in the μ *vs.* D plane, the boundary is given by

$$\mu D = \left(\sqrt{\frac{2}{1+\sigma}} - 1\right)^2 \tag{17}$$

and is shown by the solid curve in Figure 1. The region to the right of the curve corresponds to the homogeneous steady state. On this curve we have $p = 0$ (*i.e.* steady state) but an inhomogeneity develops with $k = k_0$ given by

$$2Dk_0^2 = \frac{1-\sigma}{1+\sigma} - \left[\left(\frac{2}{1+\sigma}\right)^{\frac{1}{2}} - 1\right]^2 = 2\left[\left(\frac{2}{1+\sigma}\right)^{\frac{1}{2}} - 1\right] \tag{18}$$

from equations (14, 17). If $\alpha^2 - 4\beta < 0$, then the growth rate is complex and an instability ($\mathrm{Re}(p) \geq 0$) develops for $\alpha \leq 0$. From equation (11) this requires

$$k^2(1+D) < \frac{1-\sigma}{1+\sigma} - \mu. \tag{19}$$

The growth rate is fastest for $k = 0$, which would be the preferred wave number in this case. Instability occurs for

$$\mu < \frac{1-\sigma}{1+\sigma} \tag{20}$$

with oscillations of frequency ω_0 given by

$$\omega_0 = \sqrt{\frac{1-\sigma}{1+\sigma}}\,. \tag{21}$$

To summarise, we find that the homogeneous steady state of equations (3, 4) is stable against time independent perturbations if $\mu D \geq \left[(\frac{2}{1+\sigma})^{\frac{1}{2}} - 1\right]^2$ and against oscillatory perturbations if $\mu \geq \frac{1-\sigma}{1+\sigma}$. On the boundary $\mu D = \left[(\frac{2}{1+\sigma})^{\frac{1}{2}} - 1\right]^2$, a steady state with finite wave number k_0, where $k_0^2 = \left[(\frac{2}{1+\sigma})^{\frac{1}{2}} - 1\right]/D$ is produced, while on the boundary $\mu = \frac{1-\sigma}{1+\sigma}$, an oscillatory state with $k = 0$ (*i.e.* homogeneous) and frequency ω_0 given by equation (21) is generated. These boundaries are shown in Figure 1 for $\sigma = 0.5$. The two boundaries intersect at $\mu = 1/3$ $D = 7 - 4\sqrt{3} \simeq 0.07$, which is a codimension two point for the system.

3 The model with varying removal rate

In this section, we consider the action of an oscillatory spatial dependence in the parameter μ, which we write as $\mu = \bar{\mu_0}(1 + \epsilon\cos(Kx))$. We will investigate the shift in the boundary for the formation of an inhomogeneous steady state. Consequently, we treat the linearized system (note that even with a spatially varying μ, the steady homogeneous state solution is $A = 1+\sigma$ and $B = (1+\sigma)^2$) as shown in equation (7) with the spatial dependence of μ appearing in the operator L of equation (8), where we now set $\frac{\partial}{\partial t} = 0$ since our interest is in the boundary, where the instability is stationary. Along the boundary, we expand the mean value $\bar{\mu_0}$ in powers of ϵ as the critical value μ_c of the mean removal rate $\bar{\mu_0}$ as

$$\mu_c = \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + ... \tag{22}$$

where $\mu_0 = \left[\sqrt{\frac{2}{1+\sigma}} - 1\right]^2/D$ and the perturbations $\delta A, \delta B$ as

$$\delta A = \delta A_0 + \epsilon\delta A_1 + \epsilon^2\delta A_2 + ... \tag{23}$$

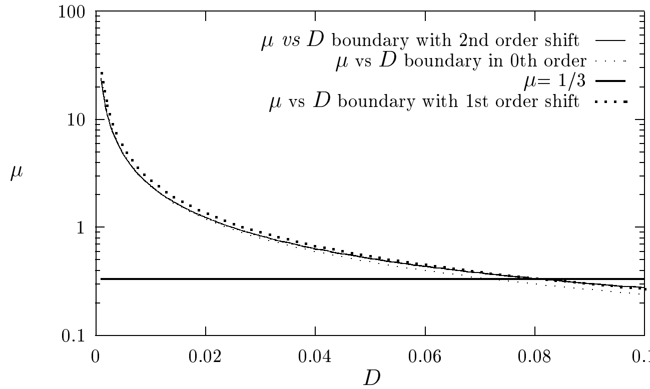

Fig. 2. The shifted μ *vs.* D phase boundaries in first and second order for $\epsilon = 1/4$ along with the boundaries in zeroth order.

$$\delta B = \delta B_0 + \epsilon\delta B_1 + \epsilon^2\delta B_2 + ... \tag{24}$$

To order unity ($\epsilon = 0$),

$$L_0 \begin{pmatrix} \delta A_0 \\ \delta B_0 \end{pmatrix} = 0 \tag{25}$$

where

$$L_0 = \begin{pmatrix} -D\nabla^2 - \frac{1-\sigma}{1+\sigma} & \frac{1}{(1+\sigma)^2} \\ 2\mu_0(1+\sigma) & \nabla^2 - \mu_0 \end{pmatrix}. \tag{26}$$

At $O(\epsilon)$

$$L_0 \begin{pmatrix} \delta A_1 \\ \delta B_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \mu_1[\delta B_0 - 2(1+\sigma)\delta A_0] + \mu_0[\delta B_0 - 2(1+\sigma)\delta A_0]\cos Kx \end{pmatrix}. \tag{27}$$

The operator L_0 has a zero eigen value for $k = k_0$ and hence equation (27) would have to be examined closely for solvability. Noting that $\delta A_0, \delta B_0 \sim \cos k_0 x$, it is clear that provided $K \neq 2k_0$, equation (27) would be solvable if $\mu_1 = 0$. For $K = 2k_0$, on the other hand, the term in μ_0 on the right hand side of equation (27) also produces a resonant term and the removal of spurious resonance requires

$$\mu_1 = -\mu_0/2. \tag{28}$$

This is the parametric resonance in the system. Our boundary is now shifted to

$$\mu_c = \mu_0(1 + \epsilon/2) \tag{29}$$

which means that the system is more susceptible to the pattern forming instability. This boundary is shown in Figure 1 for $\epsilon = 1/2$ and in Figure 2 for $\epsilon = 1/4$. For

$$\mu_2 = \frac{\mu_0}{2}\left[\frac{2[D^2\beta^2+\mu_0 D+\gamma]\alpha+2D\beta^2(\gamma-1)+4D\alpha^2+2\mu_0}{4\mu_0 D\alpha^2+2\mu_0\gamma\alpha+D^2\beta^4+2D\alpha\beta^2\gamma-2\mu_0 D\beta^2+\gamma^2\beta^2+\mu_0^2}\right]. \tag{37}$$

$$\mu = \mu_0\left[1+\frac{\epsilon^2}{2}\left(\frac{2[D^2\beta^2+\mu_0 D+\gamma]\alpha+2D\beta^2(\gamma-1)+4D\alpha^2+2\mu_0}{4\mu_0 D\alpha^2+2\mu_0\gamma\alpha+D^2\beta^4+2D\alpha\beta^2\gamma-2\mu_0 D\beta^2+\gamma^2\beta^2+\mu_0^2}\right)\right]. \tag{38}$$

$K \neq 2k_0$, equation (27) becomes

$$L_0\begin{pmatrix}\delta A_1\\ \delta B_1\end{pmatrix} = \begin{pmatrix}0\\ -\dfrac{(1+\sigma)k_0^2\mu_0}{k_0^2+\mu_0}\end{pmatrix}\delta a_0(\cos(K+k_0)x+\cos(K-k_0)x) \tag{30}$$

(where the amplitude of δA_0 has been written as δa_0) leading to

$$\delta A_1 = \delta a_0\frac{k_0^2\mu_0}{k_0^2+\mu_0}\frac{1}{1+\sigma}\left[\frac{1}{\triangle_+}\cos K_+x+\frac{1}{\Delta_-}\cos K_-x\right] \tag{31}$$

and

$$\delta B_1 = -\delta a_0\frac{(1+\sigma)k_0^2\mu_0}{k_0^2+\mu_0}\times\left[\frac{DK_+^2-\frac{1-\sigma}{1+\sigma}\cos K_+x}{\Delta_+}+\frac{DK_-^2-\frac{1-\sigma}{1+\sigma}\cos K_-x}{\Delta_-}\right] \tag{32}$$

where

$$\Delta_\pm = -\left[D{K_\pm}^4+{K_\pm}^2\left(D\mu_0-\frac{1-\sigma}{1+\sigma}\right)+\mu_0\right] \tag{33}$$

with $K_\pm = K \pm k_0$. Having found the solution to $O(\epsilon)$, we now proceed to $O(\epsilon^2)$, where we find

$$L\begin{pmatrix}\delta A_2\\ \delta B_2\end{pmatrix} = \begin{pmatrix}0\\ -\dfrac{\mu_2(1+\sigma)k_0^2}{k_0^2+\mu_0}\delta a_0\cos k_0x+\mu_0[\delta B_1-2(1+\sigma)\delta A_1]\cos Kx\end{pmatrix}. \tag{34}$$

Both terms on the right hand side of equation (34) contain resonance causing terms $\cos k_0 x$ and the removal of these leads to the condition

$$\mu_2(1+\sigma) = -\frac{\mu_0}{2}\left[\frac{D(1+\sigma){K_+}^2-(1-\sigma)}{\Delta_+}+\frac{D(1+\sigma){K_-}^2-(1-\sigma)}{\Delta_-}+\frac{2}{\Delta_+}+\frac{2}{\Delta_-}\right] \tag{35}$$

or

$$\mu_2(1+\sigma) = -\frac{\mu_0}{2}(1+\sigma)\left[\frac{{K_+}^2}{\Delta_+}+\frac{{K_-}^2}{\Delta_-}+\frac{1}{\Delta_+}+\frac{1}{\Delta_-}\right] \tag{36}$$

or

See equation (37) *above.*

The resulting boundary is given by

See equation (38) *above.*

Where $\alpha = (K^2+k_0^2)$, $\beta = (K^2-k_0^2)$, and $\gamma = \left(D\mu_0 - \dfrac{1-\sigma}{1+\sigma}\right)$.

For $K = k_0$, $\sigma = 1/2$, and $\epsilon = 1/2$, this boundary is shown as the dotted curve in Figure 1. Although this value of ϵ is not particularly small, we believe this is close to the maximum value of ϵ that a perturbation expansion can reasonably handle. We also exhibit the effect of $\epsilon = 1/4$ in Figure 2.

4 Conclusions

We have studied the effect of a spatial modulation of the removal rate on a pattern forming system. In the unmodulated system the homogeneous steady state can become unstable to a spatially periodic but temporally steady or a temporally periodic but spatially homogeneous state. On modulating spatially the removal rate, we have shown that the boundary between the homogeneous steady state and the patterned steady state shifts to stabilize the patterned state and also in the situation of parametric resonance the pattern formation is aided. It is a straightforward exercise to show that the modulation stabilizes the system against the formation of a spatially homogeneous but temporally varying period. By the same token, we can show easily that for a temporal modulation of the removal rate, the boundary between the oscillatory homogeneous and the steady patterned state will shift to stabilize the homogeneous state. As for the boundary between the steady and the time varying state the boundary will shift to stabilize the time periodic state and in the case of parametric resonance also the formation of a time periodic state is favoured. In an arbitrary situation, we expect the variation in the removal rate to be random [9,10] in space and time. Since a random distribution contains all the Fourier components, we expect that the system will react strongly to the $2k_0$ and $2\omega_0$ parts and exhibit a parametric resonance controlled by the particular strength in the power spectrum.

References

1. A.M. Turing, Philos. Trans. R. Soc. London. Ser. B **237**, 37 (1952).
2. A. Gierer, H. Meinhardt, Kybernetik **12**, 30 (1972).
3. L.A. Segel, L.J. Jackson, J. Theor. Biol. **37**, 545 (1972).
4. A.J. Koch, H. Meinhardt, Rev. Mod. Phys. **66**, 1481, (1994).
5. J.K. Bhattacherjee, K. Banerjee, Phys. Rev. Lett. **51**, 2248 (1983).
6. G. Ahlers, P.C. Hohenberg, M. Lucke, Phys. Rev. A **32**, 3493-3519 (1985).
7. P. Hall, J. Fluid Mech. **67**, 29 (1975).
8. S. Fauve, K. Kumar, C. Laroche, D. Beysens, Y. Garrabos, Phys. Rev. Lett. **68**, 3160, (1992).
9. M. Lucke, F. Schank, Phys. Rev. Lett. **54**, 1465, (1985).
10. M. Lucke, *Noise in Dynamical Systems*, edited by F. Monad, P.V.E McClintock (Cambridge University Press, Cambridge, 1987).